# Numerical Solution of Stochastic Differential Equations with Jumps in Finance

A Thesis Submitted for the Degree of
Doctor of Philosophy

by

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# Certificate

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirement for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Signed ...................................

Date 3/8/2007...................

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# Contents

# Basic Notation

| | |
|---|---|
| $x^\top$ | transpose of a vector or matrix $x$; |
| $x = (x^1, \dots, x^d)^\top$ | column vector $x \in \mathbb{R}^d$ with $i$th component $x^i$; |
| $\lvert x \rvert$ | absolute value of $x$ or Euclidean norm; |
| $A = [a^{i,j}]_{i,j=1}^{k,d}$ | $(k \times d)$-matrix $A$ with $ij$th component $a^{i,j}$; |
| $\mathbb{N} = \{1, 2, \dots\}$ | set of natural numbers; |
| $\mathbb{R} = (-\infty, \infty)$ | set of real numbers; |
| $\mathbb{R}^+ = [0, \infty)$ | set of nonnegative real numbers; |
| $\mathbb{R}^d$ | $d$-dimensional Euclidean space; |
| $(a, b)$ | open interval $a < x < b$ in $\mathbb{R}$; |
| $[a, b]$ | closed interval $a \leq x \leq b$ in $\mathbb{R}$; |
| $\Omega$ | sample space; |
| $\emptyset$ | empty set; |
| $\Delta$ | time step size of a time discretization; |
| $n! = 1 \cdot 2 \cdot \ldots \cdot n$ | factorial of $n$; |
| $\binom{i}{l} = \frac{i!}{l!(i-l)!}$ | combinatorial coefficient; |
| $[a]$ | largest integer not exceeding $a \in \mathbb{R}$; |
| $(\bmod c)$ | modulo $c$; |
| $(a)^+ = \max(a, 0)$ | maximum of $a$ and 0; |

| | |
|---|---|
| $\ln(a)$ | natural logarithm of $a$; |
| i.i.d. | independent identically distributed; |
| a.s. | almost surely; |
| $f : Q_1 \to Q_2$ | function $f$ from $Q_1$ into $Q_2$; |
| $f'$ | first derivative of $f : \mathbb{R} \to \mathbb{R}$; |
| $f''$ | second derivative of $f : \mathbb{R} \to \mathbb{R}$; |
| $\frac{\partial u}{\partial x^i}$ | $i$th partial derivative of $u : \mathbb{R}^d \to \mathbb{R}$; |
| $\partial^k_{x^i} u$ or $\left(\frac{\partial}{\partial x^i}\right)^k u$ | $k$th order partial derivative of $u$ with respect to $x^i$; |
| $\mathcal{C}^k(\mathbb{R}^d, \mathbb{R})$ | set of $k$ times continuously differentiable functions; |
| $\mathcal{C}^k_P(\mathbb{R}^d, \mathbb{R})$ | set of $k$ times continuously differentiable functions which, together with their partial derivatives of order up to $k$, have polynomial growth; |
| $\mathbf{1}_A$ | indicator function for event $A$ to be true; |
| $\mathcal{N}(\cdot)$ | Gaussian distribution function; |
| $\mathcal{A}$ | collection of events, sigma-algebra; |
| $\underline{\mathcal{A}}$ | filtration; |
| $E(X)$ | expectation of $X$; |
| $E(X \mid \mathcal{A})$ | conditional expectation of $X$ under $\mathcal{A}$; |
| $P(A)$ | probability of $A$; |
| $\mathcal{B}(U)$ | smallest sigma-algebra on $U$; |
| SDE | stochastic differential equation; |

Letters such as $K$, $\widetilde{K}$, $C$, $\widetilde{C}$, ... represent finite positive real constants that can vary from line to line. All these constants are assumed to be independent of the time step size $\Delta$. The remaining notation is either standard or will be introduced when used.

# Abstract

This thesis concerns the design and analysis of new discrete time approximations for stochastic differential equations (SDEs) driven by Wiener processes and Poisson random measures. In financial modelling, SDEs with jumps are often used to describe the dynamics of state variables such as credit ratings, stock indices, interest rates, exchange rates and electricity prices. The jump component can capture event-driven uncertainties, such as corporate defaults, operational failures or central bank announcements. The thesis proposes new, efficient, and numerically stable strong and weak approximations. Strong approximations provide efficient tools for problems such as filtering, scenario analysis and hedge simulation, while weak approximations are useful for handling problems such as derivative pricing, the evaluation of moments, and the computation of risk measures and expected utilities. The discrete time approximations proposed are divided into regular and jump-adapted schemes. Regular schemes employ time discretizations that do not include the jump times of the Poisson measure. Jump-adapted time discretizations, on the other hand, include these jump times.

The first part of the thesis introduces stochastic expansions for jump diffusions and proves new, powerful lemmas providing moment estimates of multiple stochastic integrals. The second part presents strong approximations with a new strong convergence theorem for higher order general approximations. Innovative strong derivative-free and predictor-corrector schemes are derived. Furthermore, the strong convergence of higher order schemes for pure jump SDEs is established under conditions weaker than those required for jump diffusions. The final part of the thesis presents a weak convergence theorem for jump-adapted higher order general approximations. These approximations include new derivative-free, predictor-corrector, and simplified schemes. Finally, highly efficient implementations of simplified weak schemes based on random bit generators and hardware accelerators are developed and tested.